Schur's Q-functions and Plethysm Stability

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Symmetric Functions

Definition

A *symmetric function* is a function in the variables $A = \{a_1, a_2, \dots\}$ that is invariant under any permutation of the variables.

- Example: $f(a_1, a_2) = a_1^2 a_2 + a_1 a_2^2 = f(a_2, a_1)$
- Non-example: $g(a_1, a_2) = a_1^2 a_2$, but $g(a_2, a_1) = a_1 a_2^2$
- Let $\Lambda_{\mathbb{Q}}$ denote the *ring of symmetric functions* with rational coefficients in the variables $A = \{a_1, a_2, \dots\}$
- There are several algebraic generating sets for symmetric functions

$$\begin{aligned}\Lambda_{\mathbb{Q}} &= \mathbb{Q}[e_1, e_2, e_3, \dots] \\ &= \mathbb{Q}[h_1, h_2, h_3, \dots] \\ &= \mathbb{Q}[p_1, p_2, p_3, \dots]\end{aligned}$$

Algebraic Generating Sets

Definition

The *elementary symmetric function* e_n is defined

$$e_n := \sum_{i_1 < i_2 < \cdots < i_n} a_{i_1} a_{i_2} \cdots a_{i_n} \qquad (n \geq 1)$$

The *complete (homogeneous) symmetric function* h_n is defined

$$h_n := \sum_{i_1 \leq i_2 \leq \cdots \leq i_n} a_{i_1} a_{i_2} \cdots a_{i_n} \qquad (n \geq 1)$$

The *power sum symmetric function* p_n is defined

$$p_n := \sum_{i \geq 1} a_i^n \qquad (n \geq 1)$$

- $e_0 = h_0 = p_0 = 1, \qquad e_1 = h_1 = p_1 = a_1 + a_2 + a_3 + \cdots$

Partitions

- The bases are indexed by partitions

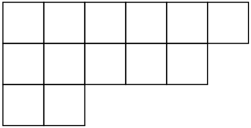
Definition

A *composition* λ is a finite sequence of integers $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{Z}^n$. It is a *partition* if its parts are non-negative and weakly decreasing,

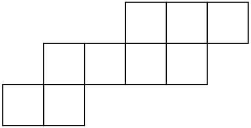
$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0.$$

- $\ell(\lambda) :=$ number of nonzero parts of λ
- The *Young Diagram* of shape

$\lambda = (6, 5, 2)$



$\lambda - \mu = (6, 5, 2) - (3, 1)$



The Bases

Definition

For a partition λ, define

$$e_\lambda := e_{\lambda_1} e_{\lambda_2} e_{\lambda_3} \cdots , \qquad h_\lambda := h_{\lambda_1} h_{\lambda_2} h_{\lambda_3} \cdots , \qquad p_\lambda := p_{\lambda_1} p_{\lambda_2} p_{\lambda_3} \cdots .$$

- The sets $\{e_\lambda\}$, $\{h_\lambda\}$, $\{p_\lambda\}$, indexed by partitions λ, are bases of $\Lambda_{\mathbb{Q}}$

Example

Let $F = h_5 h_3 h_1 + 3h_6 h_2 + 5$, then $F = h_{(5,3,1)} + 3h_{(6,2)} + 5h_{(0)}$

- Also bases: $\{m_\lambda\}$ and $\{S_\lambda\}$

Schur Functions

Theorem (Jacobi-Trudi Identity)

For a partition $\lambda \in \mathbb{Z}^n$, we have

$$S_\lambda = \det\left(h_{\lambda_i - i + j}\right)_{1 \le i,j \le n}$$

Example

Let $\lambda = (6,5,2)$, then

$$S_{(6,5,2)} = \det\begin{pmatrix} h_{6-1+1} & h_{6-1+2} & h_{6-1+3} \\ h_{5-2+1} & h_{5-2+2} & h_{5-2+3} \\ h_{2-3+1} & h_{2-3+2} & h_{2-3+3} \end{pmatrix}$$

$$= \det\begin{pmatrix} h_6 & h_7 & h_8 \\ h_4 & h_5 & h_6 \\ h_0 & h_1 & h_2 \end{pmatrix}$$

$$= h_6 h_5 h_2 - h_6^2 h_1 - h_7 h_4 h_2 + h_7 h_6 + h_8 h_4 h_1 - h_8 h_5$$

The Functions q_n

Definition (q_n's)

The functions q_n are defined by the generating function

$$\kappa_z := \prod_{a \in A} \frac{1 + az}{1 - az} = \sum_{n \in \mathbb{Z}} q_n(A) z^n.$$

Example (First Few q_n's)

$$q_1 = 2a_1 + 2a_2 + 2a_3 + \cdots = 2p_1$$
$$q_2 = 2a_1^2 + 2a_2^2 + \cdots + 2a_1 a_2 + 2a_1 a_3 + \cdots = 2p_1^2$$
$$q_3 = \frac{4}{3} p_1^3 + \frac{2}{3} p_3$$

- We work in the subring $\Gamma_{\mathbb{Q}} := \mathbb{Q}[q_1, q_2, q_3, \ldots] \subset \Lambda_{\mathbb{Q}}$
- Note: $\Gamma_{\mathbb{Q}} = \mathbb{Q}[q_1, q_3, q_5, \ldots] = \mathbb{Q}[p_1, p_3, p_5, \ldots]$

Schur's Q-functions

Definition (Schur's Q-function Q_λ)

For $r, s \in \mathbb{Z}$, define

$$Q_{(r,s)} := q_r q_s + 2\sum_{i=1}^{s}(-1)^i q_{r+i} q_{s-i}.$$

Then, for $\lambda = (\lambda_1, \ldots, \lambda_{2n})$, define

$$Q_\lambda := \operatorname{Pf} M(\lambda),$$

where

$$M(\lambda)_{ij} := \begin{cases} Q_{(\lambda_i,\lambda_j)}(A) & \text{if } i > j, \\ 0 & \text{if } i = j, \\ -Q_{(\lambda_j,\lambda_i)}(A) & \text{if } i < j, \end{cases}$$

and where $\det M = (\operatorname{Pf} M)^2$ for a skew-symmetric matrix.

Schur's Q-function Properties

Example (The matrix $M(\lambda 0)$)

Let $\lambda = (5,2,1)$. Since $\ell(\lambda)$ is odd, use $\lambda 0 = (5,2,1,0)$. Then

$$Q_{(5,2,1)} = \mathrm{Pf}\begin{pmatrix} 0 & Q_{(5,2)} & Q_{(5,1)} & Q_{(5,0)} \\ -Q_{(5,2)} & 0 & Q_{(2,1)} & Q_{(2,0)} \\ -Q_{(5,1)} & -Q_{(2,1)} & 0 & Q_{(1,0)} \\ -Q_{(5,0)} & -Q_{(2,0)} & -Q_{(1,0)} & 0 \end{pmatrix}$$

$$= Q_{(5,2)}Q_{(1,0)} - Q_{(5,1)}Q_{(2,0)} + Q_{(5,0)}Q_{(2,1)}$$

$$= q_1 q_2 q_5 - 2q_3 q_5 - 2q_2 q_6 + 2q_1 q_7$$

- This extends Q_λ to compositions λ with negative parts
  - For S_λ, Jacobi-Trudi works for compositions λ with negative parts
- The $\{Q_\lambda\}$, indexed by *strict* partitions, are a basis of $\Gamma_{\mathbb{Q}}$
- Analogous roles/properties to Schur functions S_λ
- Goal: Prove plethysm stability of Q_λ using vertex operators

Basic Properties and Question

Properties:

- Anti-symmetry:

$$Q_{(r,s)} = -Q_{(s,r)} \qquad (r+s \neq 0)$$

- Non anti-symmetry: $Q_{(0,0)} = 1$, and

$$Q_{(r,-r)} = 0, \qquad Q_{(-r,r)} = (-1)^r 2 \qquad (r \geq 1)$$

- Appending 0's: $Q_{(r,0)} = Q_{(r)} = q_r$
- $Q_\lambda = 0$ if λ is not strict

Question: How do we interpret negative parts?

Negative Parts

- $p\lambda := (p, \lambda_1, \ldots, \lambda_n)$

Proposition (G.-Jing (2025))

Let $p \in \mathbb{Z}$, $p > 0$, be a positive integer and let $\lambda \in \mathbb{Z}^n$ be a strict partition, then

$$Q_{(-p)\lambda} = \begin{cases} (-1)^{p+i+1} 2 Q_{\lambda \setminus \{\lambda_i\}} & \text{if } p = \lambda_i \text{ for some } i, \\ 0 & \text{otherwise.} \end{cases}$$

- $Q_{(-4,5,4,2)} = -2Q_{(5,2)}, \qquad Q_{(-4,5,3,2)} = 0$
- $Q_{(-p)\lambda}$ interpretation: negative parts remove rows from the Young diagram
- $S_{(-p)\lambda}$ interpretation: negative parts remove columns

Vertex Operator to Schur's Q-functions

- The vertex operator $Y(z)$ is defined

$$Y(z) := \exp\left(\sum_{\substack{n\geq 1\\ n \text{ odd}}} \frac{2}{n} p_n z^n\right) \exp\left(-\sum_{\substack{n\geq 1\\ n \text{ odd}}} \frac{\partial}{\partial p_n} z^{-n}\right)$$

- Its homogeneous components Y_n are defined

$$Y(z) = \sum_{n\in\mathbb{Z}} Y_n z^n$$

- The operator $Y_{\lambda_1} Y_{\lambda_2} \cdots Y_{\lambda_k}$ corresponds to Schur's Q-function $Q_{(\lambda_1,\lambda_2,\ldots,\lambda_k)}$

Vertex Operator Identity - Symmetric Function Statement

- Define an inner product $(\cdot,\cdot)$ on $\Gamma_{\mathbb{Q}}$ such that the Q_λ form an orthogonal basis,
$$(Q_\lambda, Q_\mu) = 2^{\ell(\lambda)}\delta_{\lambda\mu}$$
- Let $F^\perp$ denote the adjoint of multiplication by $F \in \Gamma$,
$$(F^\perp G, H) = (G, FH)$$
- In our notation, the vertex operator is $\kappa_z \cdot \kappa^\perp_{-1/z}$, where
$$\kappa_z = \sum_{n\in\mathbb{Z}} q_n z^n \qquad \kappa^\perp_{-1/z} = \sum_{n\in\mathbb{Z}} (-1/z)^n q_n^\perp$$

Theorem (G.-Jing (2025))

Let λ be a partition, then we have
$$\kappa_z \cdot \kappa^\perp_{-1/z} Q_\lambda = \sum_{p\in\mathbb{Z}} Q_{p\lambda} z^p,$$
where $p\lambda := (p, \lambda_1, \dots, \lambda_n)$.

Plethysm

- Plethysm corresponds to the composition of representations
- Denoted $F \circ G$ or $F(G)$ for $F, G \in \Lambda$
- Power sums: p_m replaces variables with their mth powers

$$p_m \circ G(a_1, a_2, a_3, \ldots) = G(a_1^m, a_2^m, a_3^m, \ldots)$$

- $p_m \circ p_n = p_{mn}, \qquad p_m \circ (F + G) = (p_m \circ F) + (p_m \circ G), \qquad$ etc.

Example

$$\begin{aligned}(p_1 + p_2^2) \circ (2p_3) &= p_1 \circ (2p_3) + (p_2 \circ (2p_3))^2 \\ &= 2(p_1 \circ p_3) + 4(p_2 \circ p_3)^2 \\ &= 2p_3 + 4p_6^2\end{aligned}$$

- To compute $F \circ G$, write F, G as polynomials in $\mathbb{Q}[p_1, p_2, p_3, \ldots]$

Plethysm Stability Motivation

Example

Let's compute $Q_{(2)} \circ Q_{(p,1)}$ for $p \geq 2$:

$$
\begin{aligned}
Q_{(2)} \circ Q_{(2,1)} &= 8Q_{(4,2)} \\
Q_{(2)} \circ Q_{(3,1)} &= 8Q_{(6,2)} + 16Q_{(5,3)} + 8Q_{(5,2,1)} + 8Q_{(4,3,1)} \\
Q_{(2)} \circ Q_{(4,1)} &= 8Q_{(8,2)} + 16Q_{(7,3)} + 8Q_{(7,2,1)} + 16Q_{(6,3,1)} + \cdots \\
Q_{(2)} \circ Q_{(5,1)} &= 8Q_{(10,2)} + 16Q_{(9,3)} + 8Q_{(9,2,1)} + 16Q_{(8,3,1)} + \cdots \\
Q_{(2)} \circ Q_{(6,1)} &= 8Q_{(12,2)} + 16Q_{(11,3)} + 8Q_{(11,2,1)} + 16Q_{(10,3,1)} + \cdots
\end{aligned}
$$

The coefficients to $Q_{(s,3,1)}$ stabilize to 16:

$$0, \quad 8, \quad 16, \quad 16, \quad 16, \quad \cdots$$

The inner product isolates these coefficients:

$$\left(Q_{(2)} \circ Q_{(p,1)}, Q_{(s,3,1)}\right) \qquad \text{for } p \in \mathbb{Z},\ 2 \cdot (p+1) = s + 3 + 1$$

Plethysm Stability Theorems

Theorem (G.-Jing (+2025))

Let λ, μ, ν be partitions, then the following sequences stabilize:

$$(Q_\lambda \circ Q_{p\mu}, Q_{s\nu}) \qquad \text{for } p \in \mathbb{Z},\ s = |\lambda|(|\mu| + p) - |\nu|,$$
$$(Q_{p\lambda} \circ Q_{\mu}, Q_{s\nu}) \qquad \text{for } p \in \mathbb{Z},\ s = (|\lambda| + p)|\mu| - |\nu|,\ \ell(\mu) > 1.$$

The following sequence increases linearly for large enough p:

$$(Q_{p\lambda} \circ Q_{(m)}, Q_{s\nu}) \qquad \text{for } p \in \mathbb{Z},\ s = (|\lambda| + p)m - |\nu|.$$

Theorem (Carré -Thibon)

Let λ, μ, ν be partitions, then the following sequences stabilize:

$$(S_\lambda \circ S_{p\mu}, S_{s\nu}) \qquad \text{for } p \in \mathbb{Z},\ s = |\lambda|(|\mu| + p) - |\nu|,$$
$$(S_{p\lambda} \circ S_{\mu}, S_{s\nu}) \qquad \text{for } p \in \mathbb{Z},\ s = (|\lambda| + p)|\mu| - |\nu|.$$

Vertex Operators and Plethysm Stability

- How does the vertex operator relate to plethysm stability? Write the sequence as a power series:

$$\sum_{p,s\in\mathbb{Z}} (Q_\lambda \circ Q_{p\mu}, Q_{s\nu})\, z^s = \sum_{p\in\mathbb{Z}} \left(Q_\lambda \circ Q_{p\mu}, \sum_{s\in\mathbb{Z}} Q_{s\nu} z^s \right)$$
$$= \sum_{p\in\mathbb{Z}} \left(Q_\lambda \circ Q_{p\mu}, \kappa_z \cdot \kappa^{\perp}_{-1/z} Q_\nu \right)$$

- Carré and Thibon used vertex operators to show plethysm stability of Schur functions
- Why the difference in stability between Schur and Schur's Q?

$$S_p(S_m(z)) = z^{mp},$$
$$Q_p(Q_m(z)) = 4pz^{mp}.$$

Future Work

- The Hall-Littlewood functions $Q_\lambda(A;t)$ generalize S_λ and Q_λ

$$Q_\lambda(A;0) = S_\lambda(A) \qquad Q_\lambda(A;-1) = Q_\lambda(A)$$

- Can we prove plethysm stability for Hall-Littlewood functions?

Thank you!