Symmetric Functions and Plethysm

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Symmetric Functions

Definition

A *symmetric polynomial* is a polynomial in the variables $a_1, \ldots, a_n$ that is invariant under any permutation of the variables.

- Example: $f(a_1, a_2) = a_1^2 a_2 + a_1 a_2^2 = f(a_2, a_1)$
- Non-example: $g(a_1, a_2) = a_1^2 a_2$, but $g(a_2, a_1) = a_1 a_2^2$
- An *alphabet* $A = \{a_1, a_2, a_3, \ldots\}$ is a set of variables (finite or infinite)

Definition

A *symmetric function* is a symmetric polynomial in infinitely-many variables $A = \{a_1, a_2, a_3, \ldots\}$ (more accurately, a it is a formal power series in the variables $A = \{a_1, a_2, a_3, \ldots\}$).

- Example: $f(A) = \sum_{a \in A} a = a_1 + a_2 + a_3 + \cdots$

The Bases

- $\Lambda_{\mathbb{Q}}$ denotes the *ring of symmetric functions* with rational coefficients
- A *partition* $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n$ is a sequence $\lambda_1 \geq \cdots \geq \lambda_n \geq 0$

Definition

The *power sum symmetric function* p_r $(r \geq 0)$ is defined $p_0 = 1$ and

$$p_r = \sum_{i \geq 1} a_i^r \quad (r \geq 1)$$

$$p_\lambda = p_{\lambda_1} p_{\lambda_2} \cdots p_{\lambda_n}.$$

- Example: $p_2(a_1, a_2, a_3) = a_1^2 + a_2^2 + a_3^2$
- Example: $p_{(2,1)}(a_1, a_2, a_3) = p_2 p_1 = (a_1^2 + a_2^2 + a_3^2)(a_1 + a_2 + a_3)$
- $\Lambda_{\mathbb{Q}} = \mathbb{Q}[p_1, p_2, \ldots] = \mathbb{Q}[e_1, e_2, \ldots] = \mathbb{Q}[h_1, h_2, \ldots]$
- $\{p_\lambda\}$, $\{e_\lambda\}$, $\{h_\lambda\}$ are vector space bases of $\Lambda_{\mathbb{Q}}$
- Another basis is the *Schur functions* $\{S_\lambda\}$

Schur Functions

Theorem (Jacobi-Trudi Identity)

For a partition $\lambda \in \mathbb{Z}^n$, we have

$$S_\lambda = \det\left(h_{\lambda_i - i + j}\right)_{1 \le i,j \le n}$$

Example

Let $\lambda = (6,5,2)$, then

$$S_{(6,5,2)} = \det\begin{pmatrix} h_{6-1+1} & h_{6-1+2} & h_{6-1+3} \\ h_{5-2+1} & h_{5-2+2} & h_{5-2+3} \\ h_{2-3+1} & h_{2-3+2} & h_{2-3+3} \end{pmatrix}$$

$$= \det\begin{pmatrix} h_6 & h_7 & h_8 \\ h_4 & h_5 & h_6 \\ h_0 & h_1 & h_2 \end{pmatrix}$$

$$= h_6 h_5 h_2 - h_6^2 h_1 - h_7 h_4 h_2 + h_7 h_6 + h_8 h_4 h_1 - h_8 h_5$$

The Functions q_n

- An uncommon basis: Schur's Q-functions

Definition

The functions $q_n(A)$ are defined by the generating series

$$\kappa_z(A) := \prod_{a \in A} \frac{1 + az}{1 - az} = \sum_{n \in \mathbb{Z}} q_n(A) z^n.$$

- $q_1 = 2a_1 + 2a_2 + 2a_3 + \cdots = 2p_1$
- $q_2 = 2a_1^2 + 2a_2^2 + \cdots + 2a_1 a_2 + 2a_1 a_3 + \cdots = 2p_1^2$
- $q_3 = \frac{4}{3} p_1^3 + \frac{2}{3} p_3$

The Subring Γ

- We will work in the subring $\Gamma := \mathbb{Z}[q_1, q_2, \ldots]$
- *Schur's Q-functions* Q_λ form a vector space basis of Γ, indexed by strict partitions λ

Partition definitions:

- A partition $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n$ is *strict* if it has no repeated nonzero parts
- Its *length* $\ell(\lambda)$ is the number of nonzero parts
- Its *weight* $|\lambda|$ is the sum of its parts.

Operations on partitions:

- Append integer $p \in \mathbb{Z}$ to beginning: $p\lambda := (p, \lambda_1, \ldots, \lambda_n)$
- Append 0 to end: $\lambda 0 := (\lambda_1, \ldots, \lambda_n, 0)$
- Remove ith part: $\lambda \setminus \{\lambda_i\} := (\lambda_1, \ldots, \lambda_{i-1}, \lambda_{i+1}, \ldots, \lambda_n)$

Pfaffian Reformulation

Definition ([GJ24a])

For $r, s \in \mathbb{Z}$, define

$$Q_{(r,s)} := q_r q_s + 2 \sum_{i=1}^{s} (-1)^i q_{r+i} q_{s-i}.$$

Then, for $\lambda = (\lambda_1, \ldots, \lambda_{2n})$, define

$$Q_\lambda := \operatorname{Pf} M(\lambda),$$

where

$$M(\lambda)_{ij} := \begin{cases} Q_{(\lambda_i,\lambda_j)}(A) & \text{if } i > j, \\ 0 & \text{if } i = j, \\ -Q_{(\lambda_j,\lambda_i)}(A) & \text{if } i < j, \end{cases}$$

and where $\det M = (\operatorname{Pf} M)^2$ for a skew-symmetric matrix ($M^t = -M$).

Basic Properties

Example

Let $\lambda = (5, 2, 1)$. Since $\ell(\lambda)$ is odd, use $\lambda 0 = (5, 2, 1, 0)$. Then

$$\begin{aligned} Q_{(5,2,1)} &= \mathrm{Pf}\begin{pmatrix} 0 & Q_{(5,2)} & Q_{(5,1)} & Q_{(5,0)} \\ -Q_{(5,2)} & 0 & Q_{(2,1)} & Q_{(2,0)} \\ -Q_{(5,1)} & -Q_{(2,1)} & 0 & Q_{(1,0)} \\ -Q_{(5,0)} & -Q_{(2,0)} & -Q_{(1,0)} & 0 \end{pmatrix} \\ &= Q_{(5,2)}Q_{(1,0)} - Q_{(5,1)}Q_{(2,0)} + Q_{(5,0)}Q_{(2,1)} \\ &= q_1q_2q_5 - 2q_3q_5 - 2q_2q_6 + 2q_1q_7 \end{aligned}$$

- Antisymmetry: $Q_{(r,s)} = -Q_{(s,r)}$ for all $r + s \neq 0$
- Non antisymmetry:
  - $Q_{(0,0)} = 1$
  - $Q_{(r,-r)} = 0$, but $Q_{(-r,r)} = (-1)^r 2$ for all $r \geq 1$
- $Q_\lambda = 0$ if λ is not strict

Negative Parts

Proposition ([GJ24a])

Let $p \in \mathbb{Z}$, $p > 0$, be a positive integer and let $\lambda \in \mathbb{Z}^n$ be a strict partition, then

$$Q_{(-p)\lambda} = (-1)^{p+\mathrm{ind}(\lambda,p)+1} 2 Q_{\lambda\setminus\{p\}},$$

where

$$\mathrm{ind}(\lambda, p) = \begin{cases} i & \text{if } p = \lambda_i, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$Q_{\lambda\setminus\{p\}} := \begin{cases} Q_{\lambda\setminus\{\lambda_i\}} & \text{if } p = \lambda_i, \\ 0 & \text{otherwise.} \end{cases}$$

- $Q_{(-4,5,4,2)} = (-1)^{4+2+1} 2 Q_{(5,2)}$
- $Q_{(-4,5,3,2)} = 0$

Inner Product and Adjoints

- Define an inner product $(\cdot, \cdot)$ on $\mathbb{C}[z] \otimes \Gamma$ such that the Q_λ form an orthogonal basis,

$$(Q_\lambda, Q_\mu) = 2^{\ell(\lambda)} \delta_{\lambda\mu}$$

- For partitions λ, μ, define the *skew Schur's Q-function* $Q_{\lambda/\mu}$ by

$$(Q_{\lambda/\mu}, F) = \left(Q_\lambda, 2^{-\ell(\mu)} Q_\mu F\right), \qquad \text{for all } F \in \Gamma$$

  - $Q_{\lambda/\mu}$ can be computed with a similar Pfaffian formula
- $F^\perp$ denotes the adjoint of multiplication by $F \in \Gamma$,

$$(FG, H) = (G, F^\perp H), \qquad \text{for all } G, H \in \Gamma$$

Vertex Operator Identity

- A *vertex operator* is a type of differential operator
- In terms of symmetric functions, the vertex operator is

$$\kappa_z \cdot \kappa^{\perp}_{-1/z}$$

where

$$\kappa_z = \sum_{n \in \mathbb{Z}} q_n z^n \qquad \kappa^{\perp}_{-1/z} = \sum_{n \in \mathbb{Z}} (-1/z)^n q_n^{\perp}$$

Theorem ([GJ24a])

Let λ be a partition, then we have

$$\kappa_z \cdot \kappa^{\perp}_{-1/z} Q_\lambda = \sum_{p \in \mathbb{Z}} Q_{p\lambda} z^p,$$

where $p\lambda := (p, \lambda_1, \ldots, \lambda_n)$.

$$\kappa_z \cdot \kappa^{\perp}_{-1/z} Q_\lambda = \sum_{p\in\mathbb{Z}} Q_{p\lambda} z^p$$

Example

Consider $p(5,2,1)$ where $p=-1$.

$$Q_{(-1,5,2,1)} = q_{-1} q_0^{\perp} Q_{(5,2,1)} - q_0 q_1^{\perp} Q_{(5,2,1)} + q_1 q_2^{\perp} Q_{(5,2,1)} - \cdots$$

Note that $q_r^{\perp} Q_\lambda = Q_{\lambda/(r)}$, so

$$Q_{(-1,5,2,1)} = -2 Q_{(5,2,1)/(1)} + 2 q_1 Q_{(5,2,1)/(2)} - \cdots$$

Since $Q_{(-1,5,2,1)} = -2 Q_{(5,2)}$ we have

$$Q_{(5,2)} = Q_{(5,2,1)/(1)} - q_1 Q_{(5,2,1)/(2)} + q_2 Q_{(5,2,1)/(3)} - \cdots .$$

Plethysm

- *Plethysm* is a type of composition, $F(G)$ or $F \circ G$
- Power sum basis works well: $p_m(p_n) = p_{mn} = p_n(p_m)$

Proof.

Recall that $p_n(A) = a_1^n + a_2^n + a_3^n + \cdots$. To compute $p_m(p_n)$, replace each variable of $p_n(A)$ with its mth power:

$$\begin{aligned} p_m(p_n) &= (a_1^m)^n + (a_2^m)^n + (a_3^m)^n + \cdots \\ &= a_1^{mn} + a_2^{mn} + a_3^{mn} + \cdots \\ &= p_{mn}(A) \end{aligned}$$

□

- Compute $p_m(G)$ by replacing every variable in G by its mth power
- $p_1(G) = G(A) = G(p_1)$, so $p_1 = a_1 + a_2 + a_3 + \cdots$ is the unit
- $G(a_1 + a_2 + \cdots) = G(A)$, so identify $A = a_1 + a_2 + a_3 + \cdots$

Plethysm in Γ

Some properties in Γ :

- $q_n(z) = 2(z)^n \quad (n \geq 1)$
- $Q_\lambda(A+B) = \sum_\mu Q_{\lambda/\mu}(A) Q_\mu(B)$
- $Q_\lambda(zA) = z^{|\lambda|} Q_\lambda(A)$
- $F(A+z) = \kappa_z^\perp F(A)$ for all $F \in \Gamma$

Sequences of plethysm:

- It's natural to consider sequences of plethysm arising from representation theory:
  $(Q_\lambda \circ Q_{p\mu}, Q_{s\nu})$ for $p \in \mathbb{Z}$ $(s = |\lambda|(|\mu| + p) - |\nu|)$

Plethysm Stability

Theorem ([GJ24b])

Let λ, μ, ν be partitions, then the sequences

$$\begin{aligned} &(Q_\lambda \circ Q_{p\mu}, Q_{s\nu}) \qquad (p \in \mathbb{Z}, s = |\lambda|(|\mu| + p) - |\nu|), \\ &(Q_{p\lambda} \circ Q_\mu, Q_{s\nu}) \qquad (p \in \mathbb{Z}, s = (|\lambda| + p)|\mu| - |\nu|) \end{aligned}$$

stabilize.

- As a power series, the first sequence says

$$\sum_{s\in\mathbb{Z}} \left(\sum_{p\in\mathbb{Z}} Q_\lambda \circ Q_{p\mu}, Q_{s\nu} \right) z^s = L(z) + \frac{cz^h}{1 - z^{|\lambda|}},$$

where $h \in \mathbb{Z}$, and L is a Laurent polynomial.

- Idea: $\sum_{s\in\mathbb{Z}} Q_{s\nu} z^s = \kappa_z \kappa^\perp_{-1/z} Q_\nu$

A Generalization - Hall-Littlewood Functions

- The *Hall-Littlewood functions* $Q_\lambda(A;t)$ interpolate between Schur functions and Schur's Q-functions

$$Q_\lambda(A;-1) = Q_\lambda(A), \qquad Q_\lambda(A,0) = S_\lambda(A)$$

Open problems

- Find an algebraic formula for $Q_{\lambda/\mu}(A,t)$
- Prove the vertex operator identity for Hall-Littlewood functions

$$\alpha_z \beta^{\perp}_{-1/z} Q_\lambda(A;t) = \sum_{p\in\mathbb{Z}} Q_{p\lambda}(A;t) z^p$$

- Do Hall-Littlewood functions have plethysm stability?

References

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