Symmetric Functions, Plethysm, and Schur's Q-functions

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Symmetric Functions

Definition

A *symmetric polynomial* is a polynomial in the variables $x_1, \ldots, x_n$ that is invariant under any permutation of the variables.

- Example: $f(x_1, x_2) = x_1^2 x_2 + x_1 x_2^2 = f(x_2, x_1)$
- Non-example: $g(x_1, x_2) = x_1^2 x_2$, but $g(x_2, x_1) = x_1 x_2^2$

Definition

A *symmetric function* is a symmetric polynomial in infinitely-many variables $x_1, x_2, x_3, \ldots$ (more accurately, a it is a formal power series in the variables $x_1, x_2, x_3, \ldots$).

- Example: For $n \geq 1$, define $p_n(x_1, x_2, \ldots) = x_1^n + x_2^n + x_3^n + \cdots$

The Ring of Symmetric Functions

- Let Λ denote the *ring of symmetric functions* with integer coefficients
- There are several bases for symmetric functions

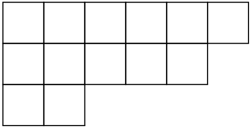
Definition

A *partition* $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{Z}^n$ is a weakly decreasing sequence of non-negative integers,

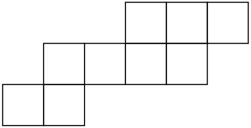
$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0.$$

- The *Young Diagram* of shape

$\lambda = (6, 5, 2)$



$\lambda - \mu = (6, 5, 2) - (3, 1)$



Definition

The *elementary symmetric function* e_r $(r \geq 0)$ is defined $e_0 = 1$ and

$$e_r = \sum_{i_1 < i_2 < \cdots < i_r} x_{i_1} x_{i_2} \cdots x_{i_r} \qquad (r \geq 1)$$

$$e_\lambda = e_{\lambda_1} e_{\lambda_2} \cdots e_{\lambda_n}.$$

- $e_1(x_1, x_2, x_3) = x_1 + x_2 + x_3$
- $e_2(x_1, x_2, x_3) = x_1x_2 + x_1x_3 + x_2x_3$

Definition

The *complete symmetric function* h_r $(r \geq 0)$ is defined $h_0 = 1$ and

$$h_r = \sum_{i_1 \leq i_2 \leq \cdots \leq i_r} x_{i_1} x_{i_2} \cdots x_{i_r} \quad (r \geq 1)$$

$$h_\lambda = h_{\lambda_1} h_{\lambda_2} \cdots h_{\lambda_n}.$$

- $h_2(x_1, x_2, x_3) = x_1x_2 + x_1x_3 + x_2x_3 + x_1^2 + x_2^2 + x_3^2$

The Bases

Definition

The *power sum symmetric function* p_r $(r \geq 0)$ is defined $p_0 = 1$ and

$$p_r = \sum_{i \geq 1} x_i^r \quad (r \geq 1)$$

$$p_\lambda = p_{\lambda_1} p_{\lambda_2} \cdots p_{\lambda_n}.$$

- Example: $p_2(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2$
- Example: $p_{(2,1)} = p_2 p_1 = (x_1^2 + x_2^2 + x_3^2)(x_1 + x_2 + x_3)$
- $\Lambda = \mathbb{Z}[e_1, e_2, \ldots] = \mathbb{Z}[h_1, h_2, \ldots]$ and $\Lambda_\mathbb{Q} = \mathbb{Q}[p_1, p_2, \ldots]$
- The e_λ and h_λ are two $\mathbb{Z}$-bases for Λ
- The p_λ are a $\mathbb{Q}$-basis for $\Lambda_\mathbb{Q}$
- Other $\mathbb{Z}$-bases are the *monomial symmetric functions* m_λ and the *Schur functions* S_λ

Schur Functions

Theorem (Jacobi-Trudi Identity)

For a partition $\lambda \in \mathbb{Z}^n$, we have

$$S_\lambda = \det\left(h_{\lambda_i - i + j}\right)_{1 \le i,j \le n}$$

Example

Let $\lambda = (6, 5, 2)$, then

$$S_{(6,5,2)} = \det\begin{pmatrix} h_{6-1+1} & h_{6-1+2} & h_{6-1+3} \\ h_{5-2+1} & h_{5-2+2} & h_{5-2+3} \\ h_{2-3+1} & h_{2-3+2} & h_{2-3+3} \end{pmatrix}$$

$$= \det\begin{pmatrix} h_6 & h_7 & h_8 \\ h_4 & h_5 & h_6 \\ h_0 & h_1 & h_2 \end{pmatrix}$$

$$= h_6 h_5 h_2 - h_6^2 h_1 - h_7 h_4 h_2 + h_7 h_6 + h_8 h_4 h_1 - h_8 h_5$$

Skew Schur Functions

- Define an inner product $\langle \cdot, \cdot \rangle$ on Λ such that Schur functions are an orthonormal basis,
$$\langle S_\lambda, S_\mu \rangle = \delta_{\lambda\mu}$$
- Define the *skew Schur function* $S_{\lambda/\mu}$ by
$$\langle S_{\lambda/\mu}, F \rangle = \langle S_\lambda, S_\mu F \rangle, \quad \forall F \in \Lambda$$
- $S_{\lambda/\mu} = \det \left(h_{\lambda_i - \mu_j - i + j} \right)_{1 \le i,j \le n}$
- Note: $S_{\lambda/0} = S_\lambda$
    - Set $\mu_j = 0$ above and get Jacobi-Trudi for S_λ
    - With Young diagrams, $\lambda - 0 = \lambda$

The Functions q_n

- An *alphabet* $A = \{a_1, a_2, \ldots\}$ is a set of variables

Definition

The functions $q_n(A)$ are defined by the generating series

$$\kappa_z(A) := \prod_{a \in A} \frac{1+az}{1-az} = \sum_{n \in \mathbb{Z}} q_n(A) z^n.$$

- If we omit variable names, then the variables are assumed to be the alphabet A, i.e., q_n means $q_n(A)$
- Note that we have $q_0 = 1$, and $q_n = 0$ for $n < 0$
- $q_1 = 2a_1 + 2a_2 + 2a_3 + \cdots = 2p_1$
- $q_2 = 2a_1^2 + 2a_2^2 + \cdots + 2a_1a_2 + 2a_1a_3 + \cdots = 2p_1^2$
- $q_3 = \frac{4}{3}p_1^3 + \frac{2}{3}p_3$

The Ring Γ

- We will work in the subring $\Gamma := \mathbb{Z}[q_1, q_2, \ldots]$
- As a vector space, Γ has a basis given by *Schur's Q-functions* Q_λ, indexed by strict partitions λ

Definition

A *composition* $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n$ is a finite sequence of integers. It is a *partition* if it is weakly decreasing and non-negative. It is *strict* if it has no repeated nonzero parts. Its *length* $\ell(\lambda)$ is the number of nonzero parts, and its *weight* $|\lambda|$ is the sum of its parts.

- Append integer $p \in \mathbb{Z}$ to beginning: $p\lambda := (p, \lambda_1, \ldots, \lambda_n)$
- Append 0 to end: $\lambda 0 := (\lambda_1, \ldots, \lambda_n, 0)$
- Remove ith part: $\lambda \setminus \{\lambda_i\} := (\lambda_1, \ldots, \lambda_{i-1}, \lambda_{i+1}, \ldots, \lambda_n)$
- Example: $5(2, -1, 0, 5)00 = (5, 2, -1, 0, 5, 0, 0)$

Pfaffian Reformulation

Definition ([GJ24a])

For $r, s \in \mathbb{Z}$, define

$$Q_{(r,s)} := q_r q_s + 2\sum_{i=1}^{s}(-1)^i q_{r+i} q_{s-i}.$$

Then, for $\lambda = (\lambda_1, \dots, \lambda_{2n})$, define

$$Q_\lambda := \operatorname{Pf} M(\lambda),$$

where

$$M(\lambda)_{ij} := \begin{cases} Q_{(\lambda_i,\lambda_j)}(A) & \text{if } i > j, \\ 0 & \text{if } i = j, \\ -Q_{(\lambda_j,\lambda_i)}(A) & \text{if } i < j, \end{cases}$$

and where $\det M = (\operatorname{Pf} M)^2$ for a skew-symmetric matrix.

Basic Properties

- $Q_{(r,s)} = -Q_{(s,r)}$ for all $r, s \geq 0$, $r + s \neq 0$
- $Q_{(r,-r)} = 0$, but $Q_{(-r,r)} = (-1)^r 2$ for all $r \geq 1$
- $Q_{(0,0)} = 1$

- $Q_{(r)} = Q_{(r,0)} = q_r$ for all $r \geq 0$
- $Q_\lambda = 0$ if λ is not strict

Example

Let $\lambda = (5,2,1)$. Since $\ell(\lambda)$ is odd, use $\lambda 0 = (5,2,1,0)$. Then

$$\begin{aligned} Q_{(5,2,1)} &= \text{Pf}\begin{pmatrix} 0 & Q_{(5,2)} & Q_{(5,1)} & Q_{(5,0)} \\ -Q_{(5,2)} & 0 & Q_{(2,1)} & Q_{(2,0)} \\ -Q_{(5,1)} & -Q_{(2,1)} & 0 & Q_{(1,0)} \\ -Q_{(5,0)} & -Q_{(2,0)} & -Q_{(1,0)} & 0 \end{pmatrix} \\ &= Q_{(5,2)}Q_{(1,0)} - Q_{(5,1)}Q_{(2,0)} + Q_{(5,0)}Q_{(2,1)} \\ &= q_1 q_2 q_5 - 2q_3 q_5 - 2q_2 q_6 + 2q_1 q_7 \end{aligned}$$

Definition Comparison

Example (Original Definition)

$$Q_{(5,2,1)} = \mathrm{Pf}\begin{pmatrix} Q_{(5,5)} & Q_{(5,2)} & Q_{(5,1)} & Q_{(5,0)} \\ Q_{(2,5)} & Q_{(2,2)} & Q_{(2,1)} & Q_{(2,0)} \\ Q_{(1,5)} & Q_{(1,2)} & Q_{(1,1)} & Q_{(1,0)} \\ Q_{(0,5)} & Q_{(0,2)} & Q_{(0,1)} & Q_{(0,0)} \end{pmatrix}$$

Example (Modified Definition)

$$Q_{(5,2,1)} = \mathrm{Pf}\begin{pmatrix} 0 & Q_{(5,2)} & Q_{(5,1)} & Q_{(5,0)} \\ -Q_{(5,2)} & 0 & Q_{(2,1)} & Q_{(2,0)} \\ -Q_{(5,1)} & -Q_{(2,1)} & 0 & Q_{(1,0)} \\ -Q_{(5,0)} & -Q_{(2,0)} & -Q_{(1,0)} & 0 \end{pmatrix}$$

Basic Consequences of New Definition

Proposition ([GJ24a])

Let $\lambda \in \mathbb{Z}^n$ be a composition, then

$$-\sum_{i=1}^{n}(-1)^i q_{\lambda_i} Q_{\lambda\setminus\{\lambda_i\}} = \begin{cases} Q_\lambda & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even.} \end{cases}$$

- Example: $\lambda = (6,0,2)$, $q_6 Q_{(0,2)} - q_0 Q_{(6,2)} + q_2 Q_{(6,0)} = Q_{(6,0,2)}$

Proposition ([GJ24a])

Let $\lambda \in \mathbb{Z}^n$ be a composition, then $Q_{\lambda 0} = Q_\lambda$.

Corollary ([GJ24a])

Let $\lambda \in \mathbb{Z}^n$ be a composition, then $Q_{\lambda/0} = Q_\lambda$.

Negative Parts

Definition

For any strict composition $\lambda \in \mathbb{Z}^n$ and any integer $p \in \mathbb{Z}$,

$$Q_{\lambda\setminus\{p\}} := \begin{cases} Q_{\lambda\setminus\{\lambda_i\}} & \text{if } p = \lambda_i, \\ 0 & \text{otherwise.} \end{cases}$$

Proposition ([GJ24a])

Let $p \in \mathbb{Z}$, $p > 0$, be a positive integer and let $\lambda \in \mathbb{Z}^n$ be a strict partition, then

$$Q_{(-p)\lambda} = (-1)^{p+\mathrm{ind}(\lambda,p)+1} 2 Q_{\lambda\setminus\{p\}},$$

where $\mathrm{ind}(\lambda, p) = i$ *if* $p = \lambda_i$, *and* 0 *otherwise.*

- $Q_{(-4,5,4,2)} = (-1)^{4+2+1} 2 Q_{(5,2)}$
- $Q_{(-4,5,3,2)} = 0$

Inner product

- Define an inner product $(\cdot,\cdot)$ on $\mathbb{C}[z]\otimes\Gamma$ such that the Q_λ form an orthogonal basis,
$$(Q_\lambda, Q_\mu) = 2^{\ell(\lambda)}\delta_{\lambda\mu}$$
- For partitions λ,μ, define the *skew Schur's Q-function* $Q_{\lambda/\mu}$ by
$$\left(Q_{\lambda/\mu}, F\right) = \left(Q_\lambda, 2^{-\ell(\mu)}Q_\mu F\right), \qquad \text{for all } F\in\Gamma$$
  - $Q_{\lambda/\mu}$ can be computed with a similar Pfaffian formula
- Let $F^\perp$ denote the adjoint of multiplication by $F\in\Gamma$ with respect to $(\cdot,\cdot)$,
$$(FQ_\lambda, Q_\mu) = (Q_\lambda, F^\perp Q_\mu)$$
- If $F=\sum_n F_n z^n$ is an infinite series, we denote $F^\perp := \sum_n z^n F_n^\perp$
  - Example: for $\kappa_z = \sum_n q_n z^n$, we have $\kappa_z^\perp = \sum_n z^n q_n^\perp$

Schur vs. Schur's Q-functions

- S_λ and Q_λ play similar roles in their rings

	Schur function S_λ	Schur's Q-function Q_λ
Basis	Orthonormal basis of Λ	Orthogonal basis of $\Gamma \subset \Lambda$
Rep. Theory	Linear rep. of $\mathfrak{S}_n$	Projective rep. of $\mathfrak{S}_n$
Boson-Fermion	Untwisted picture	Twisted picture
Combinatorics	SSYT	Shifted tableaux
Algebraic	Jacobi-Trudi (determinant)	Pfaffian formula
$Q_\lambda(A; t)$	$t = 0$	$t = -1$
Negative part	Removes column	Removes row
Vertex Op.	Yes	Yes

Vertex Operator Algebras

- Vertex operator algebras are a construction rising from Lie algebras
  - Gives basic representations of affine Kac-Moody algebras
- A *vertex operator* is a type of infinite-order differential operator in infinitely-many indeterminants
- Vertex operators are a method to realize certain symmetric functions - Schur's Q-functions, Hall-Littlewood functions, etc.
- Can be used as a tool to study symmetric functions
  - Structure of the vertex operator realization can tell us about the structure of symmetric functions

Vertex Operator to Schur's Q-functions

- The vertex operator $Y(z)$ is defined

$$Y(z) = \exp\left(\sum_{\substack{n\geq 1\\ n \text{ odd}}} \frac{2z^n}{n} H_{-n}\right) \exp\left(\sum_{\substack{n\geq 1\\ n \text{ odd}}} -\frac{2z^{-n}}{n} \frac{n}{2} \frac{\delta}{\delta_{H_{-n}}}\right)$$

where $\exp(A) = \sum_{k\geq 0} \frac{A^k}{k!}$, and the H_{-n} are indeterminates

- Its homogeneous components Y_n are defined

$$Y(z) = \sum_{n\in\mathbb{Z}} Y_n z^{-n}$$

- The operator $Y_{-\lambda_1} Y_{-\lambda_2} \cdots Y_{-\lambda_k}$ corresponds to Schur's Q-function $Q_{(\lambda_1,\lambda_2,\ldots,\lambda_k)}$

Vertex Operator Identity - Symmetric Function Statement

- In terms of symmetric functions, the vertex operator is

$$\kappa_z \cdot \kappa^{\perp}_{-1/z}$$

- Recall:

$$\kappa_z = \sum_{n\in\mathbb{Z}} q_n z^n \qquad \kappa^{\perp}_{-1/z} = \sum_{n\in\mathbb{Z}} (-1/z)^n q_n^{\perp}$$

Theorem ([GJ24a])

Let λ be a partition, then we have

$$\kappa_z \cdot \kappa^{\perp}_{-1/z} Q_\lambda = \sum_{p\in\mathbb{Z}} Q_{p\lambda} z^p,$$

where $p\lambda := (p, \lambda_1, \ldots, \lambda_n)$.

Action of the Vertex Operator

Theorem ([GJ24a])

Let λ be a partition, then we have

$$\kappa_z \cdot \kappa_{-1/z}^{\perp} Q_\lambda = \sum_{p \in \mathbb{Z}} Q_{p\lambda} z^p,$$

where $p\lambda := (p, \lambda_1, \ldots, \lambda_n)$.

- The action is just multiplication

$$\kappa_z \cdot \kappa_{-1/z}^{\perp} Q_\lambda = \cdots = \sum_{p \in \mathbb{Z}} \left(\sum_{r \geq 0} (-1)^r q_{p+r} q_r^{\perp} Q_\lambda \right) z^p$$

- Note: $q_r^{\perp} Q_\lambda = 2 Q_{\lambda/(r)}$ $(r \geq 1)$
- We go from $Q_{\lambda/(r)}$ to $Q_{p\lambda}$ by removing parts

Decomposing by Removing Parts

Lemma ([JP91])

Let $\lambda \in \mathbb{Z}^n$ be a partition, then for all positive integers $r \in \mathbb{Z}^+$ we have

$$Q_{\lambda/(r)} = -\sum_{i=1}^{n}(-1)^i q_{\lambda_i - r} Q_{\lambda \setminus \{\lambda_i\}}.$$

Lemma ([GJ24a])

Let $\lambda \in \mathbb{Z}^n$ be a partition, then for all integers $p \in \mathbb{Z}$ we have

$$Q_{p\lambda} = \begin{cases} -\sum_{i=1}^{n}(-1)^i Q_{(p,\lambda_i)} Q_{\lambda \setminus \{\lambda_i\}} & \text{if } n \text{ is odd,} \\ q_p Q_\lambda - \sum_{i=1}^{n}(-1)^i Q_{(p,\lambda_i)} Q_{\lambda \setminus \{\lambda_i\}} & \text{if } n \text{ is even.} \end{cases}$$

- Let $\lambda = (5, 2, 1)$.

Example

If $p = 4$, then

$$\begin{aligned} Q_{(4,5,2,1)} &= q_4 q_0^{\perp} Q_{(5,2,1)} - q_5 q_1^{\perp} Q_{(5,2,1)} + q_6 q_2^{\perp} Q_{(5,2,1)} - \cdots \\ &= q_4 Q_{(5,2,1)} - 2q_5 Q_{(5,2,1)/(1)} + 2q_6 Q_{(5,2,1)/(2)} - \cdots . \end{aligned}$$

Hence $Q_{(5,4,2,1)} = -q_4 Q_{(5,2,1)} + 2q_5 Q_{(5,2,1)/(1)} - 2q_6 Q_{(5,2,1)/(2)} + \cdots$.

Example

If $p = -1$, then

$$\begin{aligned} Q_{(-1,5,2,1)} &= q_{-1} q_0^{\perp} Q_{(5,2,1)} - q_0 q_1^{\perp} Q_{(5,2,1)} + q_1 q_2^{\perp} Q_{(5,2,1)} - \cdots \\ &= -2Q_{(5,2,1)/(1)} + 2q_1 Q_{(5,2,1)/(2)} - \cdots . \end{aligned}$$

But, we also know $Q_{(-1,5,2,1)} = -2Q_{(5,2)}$, and so

$$Q_{(5,2)} = Q_{(5,2,1)/(1)} - q_1 Q_{(5,2,1)/(2)} + q_2 Q_{(5,2,1)/(3)} - \cdots .$$

Plethysm

- Plethysm arises from the composition of representations
- Let $X \supseteq A$ be an alphabet
- Define $x^\lambda := x_1^{\lambda_1} x_2^{\lambda_2} \cdots$, e.g., $x^{(5,3,1)} = x_1^5 x_2^3 x_3^1$

Definition

For $P = \sum_\lambda c_\lambda x^\lambda \in \mathbb{C}[X]$, define $q_n(P)$ by the generating series

$$\kappa_z(P) = \prod_\lambda \left(\frac{1 + zx^\lambda}{1 - zx^\lambda} \right)^{c_\lambda} = \sum_{n \in \mathbb{Z}} q_n(P) z^n.$$

- Recall: any $F \in \Gamma$ can be written as a polynomial in the $q_n(A)$'s

Example

$Q_{(2,1)}(A) = q_2(A) q_1(A) - 2q_3(A)$, so $Q_{(2,1)}(p_2) = q_2(p_2) q_1(p_2) - 2q_3(p_2)$

Plethysm Properties

- Power sum basis works well: $p_m(p_n) = p_{mn} = p_n(p_m)$

Example

Recall that $p_n(A) = a_1^n + a_2^n + a_3^n + \cdots$. To compute $p_m(p_n)$, replace each variable with its mth power:

$$\begin{aligned} p_m(p_n) &= (a_1^m)^n + (a_2^m)^n + (a_3^m)^n + \cdots \\ &= a_1^{mn} + a_2^{mn} + a_3^{mn} + \cdots \\ &= p_{mn}(A) \end{aligned}$$

- Compute $p_m(g)$ by replacing every variable in g by its mth power
- $p_1(g) = g(A) = g(p_1)$, so $p_1 = a_1 + a_2 + a_3 + \cdots$ is the unit
- Identify $A = a_1 + a_2 + a_3 + \cdots$
- $S_\lambda(A+B) = \sum_\mu S_{\lambda/\mu}(A) S_\mu(B)$

Plethysm Coefficient Sequences

Some properties in Γ :

- $q_n(z) = 2(z)^n \quad (n \geq 1)$
- $Q_\lambda(A+B) = \sum_\mu Q_{\lambda/\mu}(A) Q_\mu(B)$
- $Q_\lambda(zA) = z^{|\lambda|} Q_\lambda(A)$
- $F(A+z) = \kappa_z^\perp F(A)$ for all $F \in \Gamma$

Sequences of plethysm:

- It's natural to consider sequences of plethysm arising from representation theory:
  $\left(\sum_{p \in \mathbb{Z}} Q_\lambda \circ Q_{p\mu}, Q_{s\nu} \right)$ for $s \in \mathbb{Z}$

Plethysm Stability 1

Theorem ([GJ24b])

Let λ, μ, ν *be partitions, and let* r *be the greatest integer such that* $|\lambda|(|\mu| + r) \leq |\nu|$, *then*

$$\sum_{s\in\mathbb{Z}} \left(\sum_{p\in\mathbb{Z}} Q_\lambda \circ Q_{p\mu}, Q_{s\nu} \right) z^s = L(z) + \frac{cz^h}{1 - z^{|\lambda|}},$$

where $c, h \in \mathbb{Z}$, *and* L *is a Laurent polynomial of degree at most* $|\lambda|(\mu_1 + r)$.

- Idea: $\sum_{s\in\mathbb{Z}} Q_{s\nu} z^s = \kappa_z \kappa^{\perp}_{-1/z} Q_\nu$

Plethysm Stability 2

Theorem ([GJ24b])

Let λ, μ, ν *be partitions, and let* $g(z) = \sum_{p,s\in\mathbb{Z}} \left(Q_{p\lambda} \circ Q_\mu, Q_{s\nu}\right) z^s$. *Then*

1. *If* $\ell(\mu) > 1$, *then* g *is a Laurent polynomial of degree at most* $\frac{|\nu|\cdot\mu_1}{|\mu|-\mu_1}$.
2. *If* $\mu = (m)$, *then* $g(z) = P(z) + \frac{c}{1-z^m}$, *where* $P(z)$ *is a Laurent polynomial.*

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