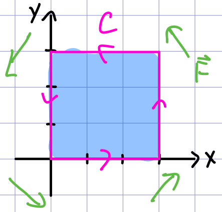
# chapter 7.3 - Green's theorem

ex: let $C$ be the boundary of $[0,1] \times [0,1]$, oriented CCW

$\vec{F}(x,y) = (-e^y + x,\ y + x^3)$. $\oint_C \vec{F} \cdot d\vec{r} = ?$




$\text{curl}_2 \vec{F} = Q_x - P_y$

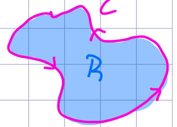
$= 3x^2 + e^y > 0 \Rightarrow$ CCW rotation

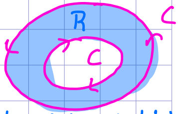
$\int_C \vec{F} \cdot d\vec{r} = +, -, 0?$

can compute directly, but would require 4 separate integrals we have found:

$$\begin{tikzcd} \mathrm{curl}_2 \vec{F} \arrow[rr, leftrightarrow, "\text{Green's Thm}"] \arrow[dr, leftrightarrow, "\text{def}"'] & & \oint_C \vec{F}\cdot d\vec{r} \\ & \text{CCW spin} \arrow[ur, leftrightarrow, "\text{above example}"'] & \end{tikzcd}$$

# Green's Theorem – ($\mathbb{R}^2$)




situation: • $R$ is a region in $\mathbb{R}^2$ w/ boundary curve $C$

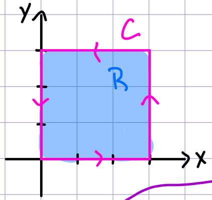
$C$ oriented s.t. $R$ is to left

• $\vec{F}$ is VF (defined on $R$ and $C$; continuously differentiable)

then: $\oint_C \vec{F} \cdot \partial \vec{r} = \iint_R \operatorname{curl}_2 \vec{F} \, \partial A$

use: turn hard line integral into easy double integral, or vice-versa ex: let $C$ be the boundary of $[0,1]\times[0,1]$, oriented CCW

$\vec{F}(x,y) = (-e^y + x \, , \, y + x^3)$. $\oint_C \vec{F}\cdot d\vec{r} = \, ?$




$$\mathrm{curl}_2 \vec{F} = 3x^2 + e^y$$

$$\oint_C \vec{F}\cdot d\vec{r} \overset{GT}{=} \iint_R \mathrm{curl}_2 \vec{F} \, dA$$

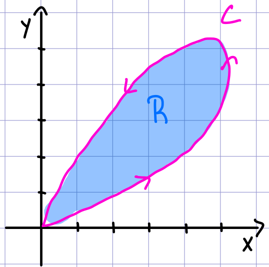
$$= \int_0^1 \int_0^1 3x^2 + e^y \, dx \, dy$$

$$= \cdots = e$$

check: orientation ✓

$\vec{F}$ defined ✓

ex: find the area enclosed by $\vec{r}(t) = \langle t - t^2, t - t^3 \rangle$, $t \in [0, 1]$




$$\text{area} = \iint_R \underbrace{1}_{\text{curl}_2 \vec{F}} \, dA$$

what is $\vec{F}$?

$$\overset{GT}{=} \oint_C \vec{F} \cdot d\vec{r}$$

$$= \oint_C \langle 0, x \rangle \cdot d\vec{r}$$

$$= \int_0^1 \underbrace{\langle 0, t - t^2 \rangle}_{\vec{F}(\vec{r}(t))} \cdot \underbrace{\langle 1 - 2t, 1 - 3t^2 \rangle}_{\vec{r}\,'(t)} \, dt$$

$$= \int_0^1 (t - t^2)(1 - 3t^2) \, dt = \cdots = 1/60$$

want $\vec{F}$ s.t. $\text{curl}_2 \vec{F} = 1$

$\Downarrow$ not conservative

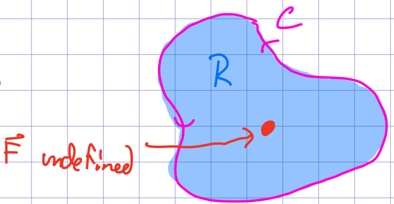
$\Rightarrow \vec{F} = \langle 0, x \rangle$ or $\langle -y, 0 \rangle$ or $\frac{1}{2} \langle -y, x \rangle$ or $\cdots$

__flawed__ logic: if $\text{curl}_2 \vec{F} = 0$ wherever $\vec{F}$ is defined

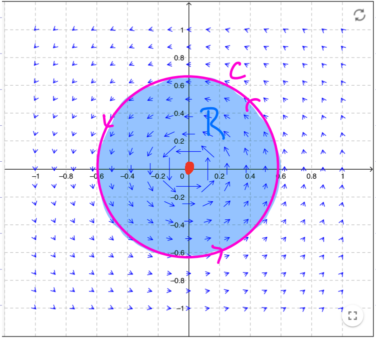
for every simple closed loop $C$, then

$$\oint_C \vec{F}\cdot d\vec{r} \overset{GT}{\underset{\text{?!}}{=}} \iint_R \text{curl}_2 \vec{F}\, dA = \iint_R 0\, dA = 0$$

GT doesn't apply if
$\vec{F}$ not defined on all of $R$



ex: $\vec{F}(x,y) = \left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle$




$\vec{F}$ not defined at $\vec{0}$

expect $\oint_C \vec{F} \cdot d\vec{r} > 0$

$\left\langle \overbrace{\frac{-y}{x^2+y^2}}^{P}, \overbrace{\frac{x}{x^2+y^2}}^{Q} \right\rangle$

a) find $\text{curl}_2 \vec{F}$

$$\text{curl}_2 \vec{F} = Q_x - P_y = \frac{y^2 - x^2}{(x^2 + y^2)^2} - \frac{y^2 - x^2}{(x^2 + y^2)^2} = \begin{cases} 0 & (x,y) \neq \vec{0} \\ \text{undef} & (x,y) = \vec{0} \end{cases}$$

b) what happens if you (incorrectly) use Green's thm to compute $\oint_C \vec{F} \cdot d\vec{r}$?

$$\oint_C \vec{F} \cdot d\vec{r} \cancel{=} \iint_R \text{curl}_2 \vec{F} \, dA = \iint_R 0 \, dA = 0$$

ex: find the area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$$A = \iint_R 1 \, dA \qquad \vec{r}(t) = \langle a \cos t, b \sin t \rangle$$

$$= \oint_C \vec{F} \cdot d\vec{r} \qquad \vec{F} = \frac{1}{2}\langle -y, x \rangle$$

$$= \int_0^{2\pi} \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t) \, dt$$

$$= \cdots$$

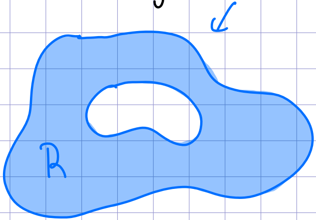
$$= \pi a b$$

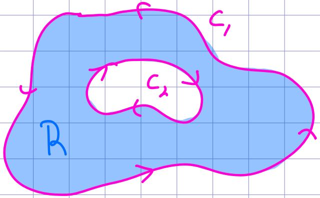
note- the two notations mean the same thing:

$$\int_c \vec{F} \cdot \partial \vec{r} = \int_c (P, Q) \cdot (\partial x, \partial y) = \int_c P \, \partial x + Q \, \partial y$$

ex: $\oint_c 3 \, \partial x + x^2 \, \partial y = \oint_c (3, x^2) \cdot \partial \vec{r}$

consider a region $R \subseteq \mathbb{R}^2$ with a hole in the middle



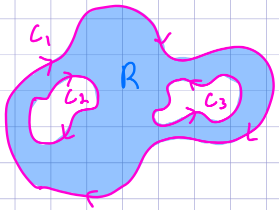

it has two <u>disconnected</u> boundary curves

(extended) Green's Theorem – $\iint_R \operatorname{curl}_2 \vec{F} \, dA = \oint_{C_1} \vec{F} \cdot d\vec{r} + \oint_{C_2} \vec{F} \cdot d\vec{r}$

ex: suppose we have

$\int_{C_1} \vec{F} \cdot \partial\vec{r} = 7$ $\int_{C_2} \vec{F} \cdot \partial\vec{r} = 4$ $\int_{C_3} \vec{F} \cdot \partial\vec{r} = 12$

find $\iint_R \text{curl}_2 \vec{F} \, \partial A$




$$\iint_R \text{curl}_2 \vec{F} \, \partial A$$

$$\overset{GT}{=} -\int_{C_1} \vec{F} \cdot \partial\vec{r} + \int_{C_2} \vec{F} \cdot \partial\vec{r}$$

$$-\int_{C_3} \vec{F} \cdot \partial\vec{r}$$

$$= -15$$